

# Covariant $(\hbar\hbar')$ -Deformed Bosonic and Fermionic Algebras as Contraction Limits of $q$ -Deformed Ones

C. Quesne<sup>1</sup>

Received January 19, 1999

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$GL_{\hbar}(n) \times GL_{\hbar'}(m)$ -covariant  $(\hbar\hbar')$ -bosonic [or  $(\hbar\hbar')$ -fermionic] algebras  $\mathcal{A}_{\hbar\hbar'\pm}(n, m)$  are built in terms of the corresponding  $R_{\hbar}$  and  $R_{\hbar'}$ -matrices by contracting the  $GL_q(n) \times GL_{q^{\pm 1}}(m)$ -covariant  $q$ -bosonic (or  $q$ -fermionic) algebras  $\mathcal{A}_{q^{\pm 1}}^{(\alpha)}(n, m)$ ,  $\alpha = 1, 2$ . When using a basis of  $\mathcal{A}_{q^{\pm 1}}^{(\alpha)}(n, m)$  wherein the annihilation operators are contragredient to the creation ones, this contraction procedure can be carried out for any  $n, m$  values. When employing instead a basis wherein the annihilation operators, like the creation ones, are irreducible tensor operators with respect to the dual quantum algebra  $U_q(\mathfrak{gl}(n)) \otimes U_{q^{\pm 1}}(\mathfrak{gl}(m))$ , a contraction limit only exists for  $n, m \in \{1, 2, 4, 6, \dots\}$ . For  $n = 2, m = 1$ , and  $n = m = 2$ , the resulting relations can be expressed in terms of coupled (anti)commutators (as in the classical case), by using  $U_{\hbar}(\mathfrak{sl}(2))$  [instead of  $\mathfrak{sl}(2)$ ] Clebsch–Gordan coefficients. Some  $U_{\hbar}(\mathfrak{sl}(2))$  rank-1/2 irreducible tensor operators recently constructed by Aizawa are shown to provide a realization of  $\mathcal{A}_{\hbar\pm}(2, 1)$ .

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## 1. INTRODUCTION

It is well known that the Lie group  $GL(2)$  admits, up to isomorphism, only two quantum group deformations with central determinant (Kupershmidt, 1992): the standard deformation  $GL_q(2)$  (Drinfeld, 1987) and the so-called Jordanian deformation  $GL_{\hbar}(2)$  (Demidov *et al.*, 1990; Zakrzewski, 1991). On the quantum algebra level, the Jordanian deformation  $U_{\hbar}(\mathfrak{sl}(2))$  of the classical enveloping algebra  $U(\mathfrak{sl}(2))$  was first considered by Ohn (1992), and its universal  $\mathcal{R}_{\hbar}$ -matrix was independently derived by Ballesteros and Herranz (1996) and by Shariati *et al.* (1996). The fundamental representation of  $U_{\hbar}(\mathfrak{sl}(2))$ , which remains undeformed, was obtained by Ohn (1992), while

<sup>1</sup>Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, B-1050 Brussels, Belgium.

the other finite-dimensional highest weight representations were first studied by Dobrev (1996). Two-parameter Jordanian deformations  $GL_{h,\alpha}(2)$ , and  $U_{h,\alpha}(\mathfrak{gl}(2))$  were also introduced by Aghamohammadi (1993), Aneva *et al.* (1997), and Parashar (1998).

Two useful tools have been devised for studying the Jordanian deformations. One is a contraction procedure that allows one to construct the latter from standard deformations (Aghamohammadi *et al.*, 1995): a similarity transformation of the defining  $R_q$  and  $T_q$ -matrices of  $GL_q(2)$  is performed using a matrix singular itself in the  $q \rightarrow 1$  limit, but in such a way that the transformed matrices are nonsingular, and yield the defining  $R_h$  and  $T_h$ -matrices of  $GL_h(2)$ .

Such a contraction technique can be generalized to higher dimensional quantum groups. It was indeed shown by Alishahiha (1995) that there exist just two independent singular maps from  $GL_q(3)$  to new quantum groups, one trivial and one nontrivial, and that the latter can be extended to  $GL_q(N)$  and  $SP_q(2N)$  for arbitrary  $N$ . This gives rise to  $GL_h(N)$  and  $SP_h(2N)$ , respectively, which are defined by their corresponding  $R_h$ -matrix.

The other tool consists of a class of nonlinear invertible maps between the generators of  $U_h(\mathfrak{sl}(2))$  and  $U(\mathfrak{sl}(2))$  (Abdesselam *et al.*, 1998b). Although there exists an equivalence relation between these maps, they may arise naturally in different contexts, and may be particularly useful for different purposes. One of them (Abdesselam *et al.*, 1996) yields an explicit and simple method for constructing the finite-dimensional irreducible representations (irreps) of  $U_h(\mathfrak{sl}(2))$ . Furthermore, it provides the decomposition rule for the tensor product of two such irreps (Aizawa, 1997), an explicit formula for  $U_h(\mathfrak{sl}(2))$  Clebsch–Gordan coefficients (CGC) (Van der Jeugt, 1998), as well as bosonic and fermionic realizations of irreducible tensor operators (ITO) for  $U_h(\mathfrak{sl}(2))$  and an extension of the Wigner–Eckart theorem to the latter (Aizawa, 1998). Another map (Abdesselam *et al.*, 1998a) provides an operational generalization of the contraction method of Aghamohammadi *et al.* (1995) and leads to the construction of  $R_h^{j_1 j_2}$  and  $T_h^{j_1 j_2}$ -matrices of arbitrary  $(j_1 \otimes j_2)$  and  $j$  irreps of  $U_h(\mathfrak{sl}(2))$ , respectively, as well as their two-parameter and/or colored extensions (Chakrabarti and Quesne, 1998). Such a technique has also been generalized to higher dimensional quantum algebras (Abdesselam *et al.*, 1997, 1998a).

In the present paper, we will apply the contraction procedure used by Alishahiha (1995) to the  $GL_q(n) \times GL_q(m)$ -covariant  $q$ -bosonic algebras  $\mathcal{A}_q^{(\alpha)}(n, m)$ ,  $\alpha = 1, 2$ , and the  $GL_q(n) \times GL_{q^{-1}}(m)$ -covariant  $q$ -fermionic ones  $\mathcal{A}_q^{(\alpha)}(n, m)$ , which were constructed some years ago by the present author (Quesne, 1993, 1994) and recently rederived by Fiore (1998) by another procedure. Such algebras generalize Pusz–Woronowicz  $GL_q(n)$ -covariant  $q$ -bosonic or  $q$ -fermionic algebras (Pusz and Woronowicz, 1989; Pusz, 1989),

$\mathcal{A}_{q^\pm}^{(\alpha)}(n)$ ,  $\alpha = 1, 2$ , to a tensor product of  $m$  Fock spaces. They are generated by  $nm$  pairs of boson- or fermion-like creation and annihilation operators  $\mathbf{A}_{is}^{\dagger}, \mathbf{A}_{is}'$  (or  $\tilde{\mathbf{A}}_{is}^{\dagger}$ ),  $i = 1, 2, \dots, n, s = 1, 2, \dots, m$ , with definite transformation properties under both  $\text{GL}_q(n)$  and  $\text{GL}_{q^\pm}(m)$ , or  $U_q(\mathfrak{gl}(n))$  and  $U_{q^\pm}(\mathfrak{gl}(m))$ .

Our purpose will be twofold. First, we will study under which conditions, if any, contracting these algebras by using two independent similarity transformations for  $\text{GL}_q(n)$  and  $\text{GL}_{q^\pm}(m)$  may lead to  $\text{GL}_h(n) \times \text{GL}_{h'}(m)$ -covariant  $(hh')$ -bosonic or  $(hh')$ -fermionic algebras  $\mathcal{A}_{hh' \pm}(n, m)$ . Second, in the  $n = 2, m = 1$ , and  $n = m = 2$  cases, we will establish some relation to the work of Aizawa (1998) on ITO and of Van der Jeugt (1998) on CGC for  $U_h(\mathfrak{sl}(2))$ .

The algebras  $\mathcal{A}_{hh' \pm}(n, m)$ , whose generators  $\mathbf{A}_{is}^{\dagger}, \mathbf{A}_{is}'$  (or  $\tilde{\mathbf{A}}_{is}$ ),  $i = 1, 2, \dots, n, s = 1, 2, \dots, m$ , have definite transformation properties under both  $\text{GL}_h(n)$  and  $\text{GL}_{h'}(m)$ , may be useful in applications of Jordanian quantum groups in various fields, such as quantum mechanics, condensed matter physics, or quantum field theory. In such applications,  $\text{GL}_h(n)$  may represent the symmetry of the physical system, while the index  $s$  may label different particles, crystal sites, or space-time points, respectively. The deformed (anti)commutation relations satisfied by  $\mathbf{A}_{is}^{\dagger}, \mathbf{A}_{is}'$  (or  $\tilde{\mathbf{A}}_{is}$ ) may then either reflect some exotic statistics or be interpreted as those of composite operators creating and annihilating some quasiparticles or dressed states of bosons (or fermions).

This paper is organized as follows. Alishahiha's contraction procedure for  $\text{GL}_h(N)$  is reviewed in Section 2, and various forms of  $\text{GL}_q(n) \times \text{GL}_{q^\pm}(m)$ -covariant  $q$ -bosonic (or  $q$ -fermionic) algebras are presented in Section 3. In Section 4, the technique of Section 2 is applied to such algebras to obtain  $\text{GL}_h(n) \times \text{GL}_{h'}(m)$ -covariant  $(hh')$ -bosonic [or  $(hh')$ -fermionic] algebras. The special cases where  $n = 2$  and  $m = 1$  or  $2$  are dealt with in Section 5. Section 6 contains the conclusion.

## 2. CONTRACTION OF $\text{GL}_q(N)$

The quantum group  $\text{GL}_q(N)$  is defined (Majid, 1990) as the associative algebra over  $\mathbb{C}$  generated by  $I$  and the noncommutative elements  $T'_{ij}$  of an  $N \times N$  matrix  $T'$  subject to the relations

$$R'_q T'_1 T'_2 = T'_2 T'_1 R'_q, \quad T'_1 = T' \otimes I, \quad T'_2 = I \otimes T' \quad (2.1)$$

where

$$R'_q = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i < j} e_{ij} \otimes e_{ji} \quad (2.2)$$

with  $i, j$  running over  $1, 2, \dots, N$ , and  $e_{ij}$  denoting the  $N \times N$  matrix with

entry 1 in row  $i$  and column  $j$ , and zeros everywhere else. It is equipped with a coproduct, a counit, and an antipode defined by

$$\Delta(T') = T'_1 \dot{\otimes} T'_2, \quad \epsilon(T') = I, \quad S(T') = T'^{-1} \quad (2.3)$$

respectively, where  $\dot{\otimes}$  denotes tensor product together with matrix multiplication. An equivalent form of the  $RTT$  relations (2.1) is obtained by replacing  $R'_q$  by  $\tau R'_q \bar{\sigma}^{-1}$ , where  $\tau$  is the twist map, i.e.,  $\tau(a \otimes b) = b \otimes a$ . Note that throughout this paper,  $q$ -deformed objects will be denoted by primed quantities, whereas unprimed ones will represent  $h$ -deformed objects.

Let us consider the similarity transformation (Aghamohammadi *et al.*, 1995; Alishahiha, 1995)

$$R''_q = (g^{-1} \otimes g^{-1}) R'_q (g \otimes g), \quad T'' = g^{-1} T' g \quad (2.4)$$

where  $g$  is the  $N \times N$  matrix defined by

$$g = \sum_i e_{ii} + \eta e_{1N}, \quad \eta = h/(q - 1) \quad (2.5)$$

Equations (2.1) and (2.3) simply become

$$R''_q T''_1 T''_2 = T''_2 T''_1 R''_q, \quad \Delta(T'') = T''_1 \dot{\otimes} T''_2, \quad \epsilon(T'') = I, \quad S(T'') = T''^{-1} \quad (2.6)$$

When  $q$  goes to one, although the parameter  $\eta$  in (2.5) becomes singular, the relations in (2.6) have a definite limit

$$R_h T_1 T_2 = T_2 T_1 R_h, \quad \Delta(T) = T_1 \dot{\otimes} T_2, \quad \epsilon(T) = I, \quad S(T) = T^{-1} \quad (2.7)$$

where  $T \equiv \lim_{q \rightarrow 1} T''$ , and

$$\begin{aligned} R_h &\equiv \lim_{q \rightarrow 1} R''_q \\ &= \sum_{ij} e_{ii} \otimes e_{jj} + h \left[ e_{11} \otimes e_{1N} - e_{1N} \otimes e_{11} + e_{1N} \otimes e_{NN} - e_{NN} \otimes e_{1N} \right. \\ &\quad \left. + 2 \sum_{i=2}^{N-1} (e_{1i} \otimes e_{iN} - e_{iN} \otimes e_{1i}) \right] + h^2 e_{1N} \otimes e_{1N} \end{aligned} \quad (2.8)$$

The resulting  $R_h$ -matrix is triangular, i.e., it is quasitriangular and  $R_h = \tau R_h^{-1} \tau$ , showing that the two equivalent forms of  $RTT$  relations for  $GL_q(N)$  actually have the same contraction limit. Together with  $I$ , the elements  $T_{ij}$  of the  $N \times N$  matrix  $T$  generate the Jordanian quantum group  $GL_h(N)$ .

### 3. COVARIANT $q$ -BOSONIC AND $q$ -FERMIONIC ALGEBRAS

Let us consider two different copies of the quantum group  $GL_q(N)$  considered in Section 2, corresponding to possibly different dimensions  $n, m$ , and parameters  $q, q^\sigma$ , respectively. Let us denote quantities referring to  $GL_q(n)$  by ordinary (primed) letters  $(R'_j, T', \dots)$ , and quantities referring to  $GL_{q^\sigma}(m)$  by script (primed) letters  $(\mathcal{R}'_{j^\sigma}, \mathcal{T}', \dots)$ . The elements  $T'_{ij}, i, j = 1, 2, \dots, n$ , of  $GL_q(n)$  are assumed to commute with the elements  $\mathcal{T}'_{st}, s, t = 1, 2, \dots, m$ , of  $GL_{q^\sigma}(m)$ . Note that for simplicity's sake, we have skipped the parameters  $q$  and  $q^\sigma$ , which should be appended to  $T'$  and  $\mathcal{T}'$ , respectively. With  $GL_q(n)$  and  $GL_{q^\sigma}(m)$ , we can associate the dual (commuting) quantum algebras  $U_q(\mathfrak{gl}(n))$  and  $U_{q^\sigma}(\mathfrak{gl}(m))$ .

Some years ago, it was shown (Quesne, 1993) that  $q$ -bosonic creation and annihilation operators  $\mathbf{A}'_{is^+}, \tilde{\mathbf{A}}'_{is}, i = 1, 2, \dots, n, s = 1, 2, \dots, m$ , that are double ITO of rank  $[1\dot{0}]_n \times [1\dot{0}]_m$  and  $[\dot{0} - 1]_n \times [\dot{0} - 1]_m$  with respect to the quantum algebra  $U_q(\mathfrak{gl}(n)) \times U_{q^\sigma}(\mathfrak{gl}(m))$ , respectively, can be constructed in terms of standard  $q$ -bosonic creation, annihilation, and number operators  $a'_{is^+}, a'_{is}, N'_{is}, i = 1, 2, \dots, n, s = 1, 2, \dots, m$  (Biedenharn, 1989; Macfarlane, 1989; Sun and Fu, 1989), acting in a tensor product Fock space  $\mathbf{F} = \prod_{i=1}^n \prod_{s=1}^m \otimes F_{is}$ . Here  $[1\dot{0}]_n$  and  $[\dot{0} - 1]_n$  denote  $n$ -row Young diagrams, the dot over 0 meaning that this numeral is repeated as often as necessary. It is straightforward to extend such a construction to covariant  $q$ -fermionic operators, provided one replaces  $U_q(\mathfrak{gl}(m))$  by  $U_{q^{-1}}(\mathfrak{gl}(m))$  and standard  $q$ -bosonic operators by standard  $q$ -fermionic ones (Chaichian and Kulish, 1990; Hayashi, 1990).

The annihilation operators  $\mathbf{A}'_{is}$ , contragredient to  $\mathbf{A}'_{is^+}$ , can also be considered, and are related to the covariant ones  $\tilde{\mathbf{A}}'_{is}$  through the equation

$$\tilde{\mathbf{A}}'_{is} = (-1)^{i+s} q^{[n-2i+1+\sigma(m-2s+1)]/2} \mathbf{A}'_{i's'} \tag{3.1}$$

where  $i' \equiv n + 1 - i, s' \equiv m + 1 - s$ , and  $\sigma = +1$  (resp.  $-1$ ) for  $q$ -bosons (resp.  $q$ -fermions). In matrix form, Eq. (3.1) can be rewritten as

$$\tilde{\mathbf{A}}' = \mathbf{A}' \mathbf{C}', \quad \mathbf{C}' = C'_q \mathcal{C}'_{q^\sigma} \tag{3.2}$$

where

$$C'_q = \sum_i (-1)^{n-i} q^{-(n-2i+1)/2} e_{ii'}, \quad \mathcal{C}'_{q^\sigma} = \sum_s (-1)^{m-s} q^{-\sigma(m-2s+1)/2} e_{ss'} \tag{3.3}$$

As happens in the  $m = 1$  case for the  $GL_q(n)$ -covariant  $q$ -bosonic or  $q$ -fermionic operators (Pusz and Woronowicz, 1989; Pusz, 1989), there actually exist two independent ways of constructing  $\mathbf{A}'_{is^+}$  and  $\tilde{\mathbf{A}}'_{is}$  (or  $\mathbf{A}'_{is}$ ) in terms of  $a'_{is^+}, a'_{is}, N'_{is}$ . According to the choice made, the operators  $\mathbf{A}'_{is^+}$  and  $\tilde{\mathbf{A}}'_{is}$ , or

$\mathbf{A}'_{is^+}$  and  $\mathbf{A}'_{is}$ , generate with  $\mathbf{I} = I\mathcal{F}$  one of two different  $U_q(\mathfrak{gl}(n)) \times U_q^\sigma(\mathfrak{gl}(m))$ -module or  $GL_q(n) \times GL_q^\sigma(m)$ -comodule algebras, which will be denoted by  $\mathcal{A}'_{q\sigma}(1)(n, m)$  and  $\mathcal{A}'_{q\sigma}(2)(n, m)$ . The defining relations of such algebras can be written in two compact forms, enhancing the transformation properties of the operators under the quantum group  $GL_q(n) \times GL_q^\sigma(m)$  or the corresponding quantum algebra  $U_q(\mathfrak{gl}(n)) \times U_q^\sigma(\mathfrak{gl}(m))$ , respectively, as well as in componentwise form using  $q$ -(anti)commutators.

In the first compact form, the defining relations of  $\mathcal{A}'_{q\sigma}(1)(n, m)$  in the  $\{\mathbf{A}'_{is^+}, \mathbf{A}'_{is}\}$  basis read (Quesne, 1994; Fiore, 1998)

$$R'_q \mathbf{A}'_1{}^+ \mathbf{A}'_2{}^+ = \sigma \mathbf{A}'_2{}^+ \mathbf{A}'_1{}^+ \mathcal{R}'_{q^\sigma} \tag{3.4}$$

$$R'_q \mathbf{A}'_2 \mathbf{A}'_1 = \sigma \mathbf{A}'_1 \mathbf{A}'_2 \mathcal{R}'_{q^\sigma} \tag{3.5}$$

$$\mathbf{A}'_2 \mathbf{A}'_1{}^+ = \mathbf{I}_{21} + \sigma R_q{}^{t_1} \mathcal{R}'_{q^\sigma}{}^{t_1} \mathbf{A}'_1{}^+ \mathbf{A}'_2 \tag{3.6}$$

while those of  $\mathcal{A}'_{q\sigma}(2)(n, m)$  are given by Eqs. (3.4) and (3.5) and

$$\mathbf{A}'_1 \mathbf{A}'_2{}^+ = \mathbf{I}_{12} + \sigma R_q{}^{t_2} \mathcal{R}'_{q^\sigma}{}^{t_2} \sigma \mathbf{A}'_2{}^+ \mathbf{A}'_1 \tag{3.7}$$

Here we use the defining  $R'_q$ -matrix of  $GL_q(n)$ , given in Eq. (2.2), and its counterpart  $\mathcal{R}'_{q^\sigma}$  for  $GL_q^\sigma(m)$ , as well as a shorthand tensor notation similar to that of Eq. (2.1), with  $t_1$  (resp.  $t_2$ ) denoting transposition in the first (resp. second) space of the tensor product.

When using instead the  $\{\mathbf{A}'_{is^+}, \tilde{\mathbf{A}}'_{is}\}$  basis of  $\mathcal{A}'_{q\sigma}(1)(n, m)$  and  $\mathcal{A}'_{q\sigma}(2)(n, m)$ , Eqs. (3.5)–(3.7) become (Quesne, 1994)

$$R'_q \tilde{\mathbf{A}}'_1 \tilde{\mathbf{A}}'_2 = \sigma \tilde{\mathbf{A}}'_2 \tilde{\mathbf{A}}'_1 \mathcal{R}'_{q^\sigma} \tag{3.8}$$

$$\tilde{\mathbf{A}}'_2 \mathbf{A}'_1{}^+ = \mathbf{C}'_{12} + \sigma \mathbf{A}'_1{}^+ \tilde{\mathbf{A}}'_2 \tilde{R}'_q{}^{-1} \tilde{\mathcal{R}}'_{q^\sigma}{}^{-1} \tag{3.9}$$

and

$$\tilde{\mathbf{A}}'_1 \mathbf{A}'_2{}^+ = \mathbf{C}'_{21} + \sigma \mathbf{A}'_2{}^+ \tilde{\mathbf{A}}'_1 \tilde{R}'_q \tilde{\mathcal{R}}'_{q^\sigma} \tag{3.10}$$

where

$$\tilde{R}'_q \equiv C'_{q,1}{}^{-1} (R_q{}^{-1}){}^{t_1} C'_{q,1} = C'_{q,2}{}^{-1} (R_q{}^{t_2}){}^{-1} C'_{q,2} \tag{3.11}$$

and similarly for  $\tilde{\mathcal{R}}'_{q^\sigma}$ . Note that one can go from  $\mathcal{A}'_{q\sigma}(1)(n, m)$  to  $\mathcal{A}'_{q\sigma}(2)(n, m)$  by making the substitutions  $R'_q \rightarrow \tau R_q{}^{-1} \tau$ ,  $\mathcal{R}'_{q^\sigma} \rightarrow \tau \mathcal{R}'_{q^\sigma}{}^{-1} \tau$ .

In either form (3.4)–(3.6) [resp. (3.4), (3.5), (3.7)] or (3.4), (3.8), (3.9) [resp. (3.4), (3.8), (3.10)], it is easy to see that  $\mathcal{A}'_{q\sigma}(1)(n, m)$  [resp.  $\mathcal{A}'_{q\sigma}(2)(n, m)$ ] is a  $GL_q(n) \times GL_q^\sigma(m)$ -comodule algebra. The transformation

$$\varphi'(\mathbf{A}'^+) = \mathbf{A}'^+ T' \mathcal{T}', \quad \varphi'(\mathbf{A}') = T'{}^{-1} \mathcal{T}'{}^{-1} \mathbf{A}' \tag{3.12}$$

or

$$\varphi'(\mathbf{A}'^+) = \mathbf{A}'^+ T' \mathcal{T}', \quad \varphi'(\tilde{\mathbf{A}}') = \tilde{\mathbf{A}}' \tilde{T}' \tilde{\mathcal{T}}' \tag{3.13}$$

where  $T'_{ij} \in \text{GL}_q(n)$ ,  $\mathcal{T}'_{st} \in \text{GL}_{q^\sigma}(m)$ ,  $\tilde{T}' = C_q^{-1}(T'^{-1})^t C'_q$ , and  $\tilde{\mathcal{T}}' = \mathcal{C}'_{q^\sigma} \mathcal{T}'^{-1} \mathcal{C}'_{q^\sigma}$ , indeed leaves the defining equations invariant, while being consistent with the  $\text{GL}_q(n) \times \text{GL}_{q^\sigma}(m)$  coalgebra structure, as given in Eq. (2.3), and its counterpart for  $\text{GL}_{q^\sigma}(m)$ .

For  $m = 1$ , one gets  $\mathcal{R}'_{q^\sigma} = q^\sigma$ ,  $\mathcal{C}'_{q^\sigma} = 1$ ,  $\tilde{\mathcal{R}}'_{q^\sigma} = q^{-\sigma}$ , so that the defining relations of  $\mathcal{A}'_{q^\sigma}(n, 1)$  and  $\mathcal{A}'_{q^\sigma}(n, 1)$  coincide with those of the two independent Pusz–Woronowicz algebras (Pusz and Woronowicz, 1989; Pusz, 1989).

The second compact form uses coupled  $q$ -(anti)commutators, defined by (Quesne, 1993)

$$\begin{aligned} & [T^{[\lambda_1]_n[\lambda'_1]_m}, U^{[\lambda_2]_n[\lambda'_2]_m}]_{\Lambda_n \Lambda'_m}^{\Lambda_n \Lambda'_m} q^\alpha \\ &= [T^{[\lambda_1]_n[\lambda'_1]_m} \times U^{[\lambda_2]_n[\lambda'_2]_m}]_{\Lambda_n \Lambda'_m}^{\Lambda_n \Lambda'_m} \\ & \quad - \sigma(-1)^\epsilon q^\alpha [U^{[\lambda_2]_n[\lambda'_2]_m} \times T^{[\lambda_1]_n[\lambda'_1]_m}]_{\Lambda_n \Lambda'_m}^{\Lambda_n \Lambda'_m} \end{aligned} \tag{3.14}$$

Here the left-hand side is a coupled  $q$ -commutator (resp.  $q$ -anticommutator) for  $\sigma = +1$  (resp.  $-1$ ),  $T^{[\lambda_1]_n[\lambda'_1]_m}$  and  $U^{[\lambda_2]_n[\lambda'_2]_m}$  denote two double ITOs of rank  $[\lambda_1]_n \times [\lambda'_1]_m$  and  $[\lambda_2]_n \times [\lambda'_2]_m$  with respect to  $U_q(\mathfrak{gl}(n)) \times U_{q^\sigma}(\mathfrak{gl}(m))$ , respectively, their tensor product of rank  $[\Lambda]_n \times [\Lambda']_m$  is defined by

$$\begin{aligned} & [T^{[\lambda_1]_n[\lambda'_1]_m} \times U^{[\lambda_2]_n[\lambda'_2]_m}]_{\Lambda_n \Lambda'_m}^{\Lambda_n \Lambda'_m} \\ &= \sum_{(\mu_1)_n(\mu'_1)_m(\mu_2)_n(\mu'_2)_m} \langle [\lambda_1]_n(\mu_1)_n, [\lambda_2]_n(\mu_2)_n | [\Lambda]_n(M)_q \rangle \\ & \quad \times \langle [\lambda'_1]_m(\mu'_1)_m, [\lambda'_2]_m(\mu'_2)_m | [\Lambda']_m(M')_q \rangle T^{[\lambda_1]_n[\lambda'_1]_m}_{(\mu_1)_n(\mu'_1)_m} U^{[\lambda_2]_n[\lambda'_2]_m}_{(\mu_2)_n(\mu'_2)_m} \end{aligned} \tag{3.15}$$

and the phase factor  $\epsilon$  is given by

$$\begin{aligned} \epsilon &= \phi([\lambda_1]_n) + \phi([\lambda_2]_n) - \phi([\Lambda]_n) + \phi([\lambda'_1]_m) \\ & \quad + \phi([\lambda'_2]_m) - \phi([\Lambda']_m) \end{aligned} \tag{3.16}$$

$$\begin{aligned} \phi([\lambda_1]_n) &= \frac{1}{2} \sum_{i=1}^n (n + 1 - 2i) \lambda_{1i}, \\ \phi([\lambda'_1]_m) &= \frac{1}{2} \sum_{s=1}^m (m + 1 - 2s) \lambda'_{1s} \end{aligned} \tag{3.17}$$

In Eq. (3.15),  $\langle \cdot, \cdot \rangle_q$  and  $\langle \cdot, \cdot \rangle_{q^\sigma}$  denote  $U_q(\mathfrak{gl}(n))$  and  $U_{q^\sigma}(\mathfrak{gl}(m))$  CGC (Biedenharn, 1990), respectively, and we have assumed that the couplings are multi-

plicity-free [which is the case for the generators of  $\mathcal{A}_{q\sigma}^{(1)}(n, m)$  and  $\mathcal{A}_{q\sigma}^{(2)}(n, m)$ ].

Such a compact form only exists for the  $\{\mathbf{A}'_{is^+}, \tilde{\mathbf{A}}'_{is}\}$  basis, since  $\mathbf{A}'_{is^+}$  and  $\tilde{\mathbf{A}}'_{is}$  (but not  $\mathbf{A}'_{is}$ ) have a definite rank with respect to  $U_q(\mathfrak{gl}(n)) \times U_{q^\sigma}(\mathfrak{gl}(m))$ , namely  $[1\dot{0}]_n \times [1\dot{0}]_m$  and  $[\dot{0} - 1]_n \times [\dot{0} - 1]_m$ , respectively. For  $\mathcal{A}_{q\sigma}^{(1)}(n, m)$ , one finds (Quesne, 1993)

$$[\mathbf{A}'^+, \mathbf{A}'^+]^{[2\dot{0}]_n[1\dot{2}\dot{0}]_m} = [\mathbf{A}'^+, \mathbf{A}'^+]^{[1\dot{2}\dot{0}]_n[2\dot{0}]_m} = 0 \tag{3.18}$$

$$[\tilde{\mathbf{A}}', \tilde{\mathbf{A}}']^{[\dot{0}-2]_n[\dot{0}(-1)^2]_m} = [\tilde{\mathbf{A}}', \tilde{\mathbf{A}}']^{[\dot{0}(-1)^2]_n[\dot{0}-2]_m} = 0 \tag{3.19}$$

in the  $q$ -bosonic case ( $\sigma = +1$ ), or

$$\{\mathbf{A}'^+, \mathbf{A}'^+\}^{[2\dot{0}]_n[2\dot{0}]_m} = \{\mathbf{A}'^+, \mathbf{A}'^+\}^{[1\dot{2}\dot{0}]_n[1\dot{2}\dot{0}]_m} = 0 \tag{3.20}$$

$$\{\tilde{\mathbf{A}}', \tilde{\mathbf{A}}'\}^{[\dot{0}-2]_n[\dot{0}-2]_m} = \{\tilde{\mathbf{A}}', \tilde{\mathbf{A}}'\}^{[\dot{0}(-1)^2]_n[\dot{0}(-1)^2]_m} = 0 \tag{3.21}$$

in the  $q$ -fermionic one ( $\sigma = -1$ ), and

$$\begin{aligned} [\tilde{\mathbf{A}}', \mathbf{A}'^+]^{[1\dot{0}-1]_n[1\dot{0}-1]_m} &= [\tilde{\mathbf{A}}', \mathbf{A}'^+]_{q^{\sigma_m-1}}^{[1\dot{0}-1]_n[\dot{0}]_m} \\ &= [\tilde{\mathbf{A}}', \mathbf{A}'^+]_{q^n}^{[\dot{0}]_n[1\dot{0}-1]_m} = 0 \end{aligned} \tag{3.22}$$

$$[\tilde{\mathbf{A}}', \mathbf{A}'^+]_{q^{n+\sigma_m}}^{[\dot{0}]_n[\dot{0}]_m} = \sqrt{[n]_q[m]_q} \mathbf{I} \tag{3.23}$$

in both cases ( $\sigma = \pm 1$ ). For simplicity's sake, we have not written the  $U_q(\mathfrak{gl}(n)) \times U_{q^\sigma}(\mathfrak{gl}(m))$  irrep row labels  $(M_1)_n(M_2)_m$ . As usual,  $q$ -numbers are defined by  $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$ . For  $\mathcal{A}_{q\sigma}^{(2)}(n, m)$ , Eqs. (3.18)–(3.23) remain valid except for the substitution  $q \rightarrow q^{-1}$  in the lower subscripts in Eqs. (3.22) and (3.23).

By using the explicit form of the  $R'_q$  and  $\mathcal{R}'_{q^\sigma}$  matrix elements given in Eq. (2.2), or the explicit values of the  $U_q(\mathfrak{gl}(n))$  and  $U_{q^\sigma}(\mathfrak{gl}(m))$  CGC (Biedenharn, 1990) together with Eq. (3.1), Eqs. (3.4)–(3.6), or (3.18)–(3.23), for  $\mathcal{A}_{q\sigma}^{(1)}(n, m)$  can be rewritten in componentwise form. The results read (Quesne, 1993)

$$\{\mathbf{A}'_{is^+}, \mathbf{A}'_{it^+}\} = 0 \tag{3.24}$$

in the  $q$ -fermionic case ( $\sigma = -1$ ), and

$$[\mathbf{A}'_{is^+}, \mathbf{A}'_{it^+}]_{q^{-1}} = 0, \quad s < t \tag{3.25}$$

$$[\mathbf{A}'_{is^+}, \mathbf{A}'_{js^+}]_{q^{-\sigma}} = 0, \quad i < j \tag{3.26}$$

$$[\mathbf{A}'_{is^+}, \mathbf{A}'_{jt^+}] = 0, \quad i > j, s < t \tag{3.27}$$

$$[\mathbf{A}'_{is^+}, \mathbf{A}'_{jt^+}] = -(q - q^{-1})\mathbf{A}'_{js^+}\mathbf{A}'_{it^+}, \quad i < j, s < t \tag{3.28}$$

$$[\mathbf{A}'_{is}, \mathbf{A}'_{jt^+}] = 0, \quad i \neq j, s \neq t \tag{3.29}$$

$$[\mathbf{A}'_{is}, \mathbf{A}'_{js^+}]_q^\sigma = (q - q^{-1}) \sum_{t=1}^{s-1} \mathbf{A}'_{jt^+} \mathbf{A}'_{it}, \quad i \neq j \quad (3.30)$$

$$[\mathbf{A}'_{is}, \mathbf{A}'_{it^+}]_q = \sigma(q - q^{-1}) \sum_{j=1}^{i-1} \mathbf{A}'_{jt^+} \mathbf{A}'_{js}, \quad s \neq t \quad (3.31)$$

$$\begin{aligned} [\mathbf{A}'_{is}, \mathbf{A}'_{is^+}]_q^{1+\sigma} &= \mathbf{I} + (q^{2\sigma} - 1) \sum_{j=1}^{i-1} \mathbf{A}'_{js^+} \mathbf{A}'_{js} + (q^2 - 1) \sum_{t=1}^{s-1} \mathbf{A}'_{it^+} \mathbf{A}'_{it} \\ &\quad + (q - q^{-1})^2 \sum_{j=1}^{i-1} \sum_{t=1}^{s-1} \mathbf{A}'_{jt^+} \mathbf{A}'_{jt} \end{aligned} \quad (3.32)$$

in both  $q$ -bosonic and  $q$ -fermionic cases ( $\sigma = \pm 1$ ), together with the Hermitian conjugates of Eqs. (3.24)–(3.28) (for real  $q$ ). Here, for  $q$ -bosons (resp.  $q$ -fermions),  $[\ , \ ]$  denotes a commutator (resp. anticommutator), and  $[\ , \ ]_{q^\alpha}$  a  $q$ -commutator (resp.  $q$ -anticommutator), i.e.,  $[A, B]_{q^\alpha} \equiv AB - \sigma q^\alpha BA$ .

For  $\mathcal{A}_{q^\sigma}^{(2)}(n, m)$ , Eqs. (3.24)–(3.29) remain unchanged, whereas Eqs. (3.30)–(3.32) are replaced by

$$[\mathbf{A}'_{is}, \mathbf{A}'_{js^+}]_q^{-\sigma} = - (q - q^{-1}) \sum_{t=s+1}^m \mathbf{A}'_{jt^+} \mathbf{A}'_{it}, \quad i \neq j \quad (3.33)$$

$$[\mathbf{A}'_{is}, \mathbf{A}'_{it^+}]_q^{-1} = -\sigma (q - q^{-1}) \sum_{j=i+1}^n \mathbf{A}'_{jt^+} \mathbf{A}'_{js}, \quad s \neq t \quad (3.34)$$

$$\begin{aligned} [\mathbf{A}'_{is}, \mathbf{A}'_{is^+}]_q^{-1-\sigma} &= \mathbf{I} + (q^{-2\sigma} - 1) \sum_{j=i+1}^n \mathbf{A}'_{js^+} \mathbf{A}'_{js} + (q^{-2} - 1) \sum_{t=s+1}^m \mathbf{A}'_{it^+} \mathbf{A}'_{it} \\ &\quad + (q - q^{-1})^2 \sum_{j=i+1}^n \sum_{t=s+1}^m \mathbf{A}'_{jt^+} \mathbf{A}'_{jt} \end{aligned} \quad (3.35)$$

Note again that for  $m = 1$ , Eqs. (3.24)–(3.35) give back the Pusz–Woronowicz results (Pusz and Woronowicz, 1989; Pusz, 1989).

#### 4. COVARIANT $(\hbar\hbar')$ -BOSONIC AND $(\hbar\hbar')$ -FERMIONIC ALGEBRAS

Let us apply the contraction procedure of Section 2 to the  $\text{GL}_q(n) \times \text{GL}_{q^\sigma}(m)$ -covariant  $q$ -bosonic (or  $q$ -fermionic) algebras  $\mathcal{A}_{q^\sigma}^{(1)}(n, m)$  and  $\mathcal{A}_{q^\sigma}^{(2)}(n, m)$ . We shall successively consider the cases where they are defined in the  $\{\mathbf{A}'_{is^+}, \mathbf{A}'_{is}\}$  basis or in the  $\{\mathbf{A}'_{is^+}, \tilde{\mathbf{A}}'_{is}\}$  one.

Since we now have two commuting copies of  $\text{GL}_q(\mathbb{N})$ , we have to consider two transformation matrices of type (2.5),  $g = \sum_i e_{ii} + \eta e_{1n}$ , and

$\mathfrak{g} = \sum_s e_{ss} + \eta' e_{1m}$ . They act on  $GL_q(n)$  and  $GL_q^\sigma(m)$ , respectively, and depend upon two parameters  $\eta \equiv h/(q - 1)$  and  $\eta' \equiv h'/(q^\sigma - 1)$ , which we may assume independent.

Let us first consider Eqs. (3.4)–(3.6), defining  $\mathcal{A}_{q\sigma}^{(1)}(n, m)$  in the  $\{\mathbf{A}'_{is^+}, \mathbf{A}'_{is}\}$  basis, and introduce transformed  $q$ -bosonic (or  $q$ -fermionic) operators  $\mathbf{A}''^+ = \mathbf{A}'^+ \mathbf{g}$ ,  $\mathbf{A}'' = \mathbf{g}^{-1} \mathbf{A}'$ , where  $\mathbf{g} = gg$ , i.e.,  $\mathbf{g}_{is, jt} = g_{ij} g_{st}$ . By using the property  $R_q^t = \tau R'_q \tau$  satisfied by (2.2) and a similar one for  $\mathcal{R}_{q^\sigma}$ , it is straightforward to show that Eqs. (3.4)–(3.6) become

$$\mathbf{A}_1''^+ \mathbf{A}_2''^+ = \sigma \mathbf{A}_2''^+ \mathbf{A}_1''^+ (\tau R_q^{\prime-1} \tau) \mathcal{R}_{q^\sigma}'' \tag{4.1}$$

$$\mathbf{A}_1'' \mathbf{A}_2'' = \sigma R_q'' (\tau \mathcal{R}_{q^\sigma}''^{-\sigma} \tau) \mathbf{A}_2'' \mathbf{A}_1'' \tag{4.2}$$

$$\mathbf{A}_2'' \mathbf{A}_1''^+ = \mathbf{I}_{21} + \sigma R_q^{\prime t1} \mathcal{R}_{q\sigma}''^t \mathbf{A}_1''^+ \mathbf{A}_2'' \tag{4.3}$$

Defining now  $(hh')$ -bosonic [or  $(hh')$ -fermionic] operators by

$$\mathbf{A}_{is}^+ \equiv \lim_{q \rightarrow 1} \mathbf{A}_{is}''^+, \quad \mathbf{A}_{is} \equiv \lim_{q \rightarrow 1} \mathbf{A}_{is}'' \tag{4.4}$$

and taking the  $q \rightarrow 1$  limit of Eqs. (4.1)–(4.3), we obtain that together with  $\mathbf{I}$ , they generate an algebra  $\mathcal{A}_{hh'\sigma}(n, m)$ , whose defining relations are

$$\mathbf{A}_1^+ \mathbf{A}_2^+ = \sigma \mathbf{A}_2^+ \mathbf{A}_1^+ R_h \mathcal{R}_{h'} \tag{4.5}$$

$$\mathbf{A}_1 \mathbf{A}_2 = \sigma R_h \mathcal{R}_{h'} \mathbf{A}_2 \mathbf{A}_1 \tag{4.6}$$

$$\mathbf{A}_2 \mathbf{A}_1^+ = \mathbf{I}_{21} + \sigma R_h^t \mathcal{R}_{h'}^t \mathbf{A}_1^+ \mathbf{A}_2 \tag{4.7}$$

In deriving the latter, we explicitly used the fact that both  $R_h$  and  $\mathcal{R}_{h'}$  are triangular. Similarly, the transformation (3.12) goes into

$$\varphi(\mathbf{A}^+) = \mathbf{A}^+ T \mathcal{T}, \quad \varphi(\mathbf{A}) = T^{-1} \mathcal{T}^{-1} \mathbf{A} \tag{4.8}$$

where  $T_{ij} \in GL_h(n)$ ,  $\mathcal{T}_{st} \in GL_{h'}(m)$ , and  $\varphi$  leaves Eqs. (4.5)–(4.7) invariant, while being consistent with the  $GL_h(n) \times GL_{h'}(m)$  coalgebra structure, as given by Eq. (2.7). Hence,  $\mathcal{A}_{hh'\sigma}(n, m)$  is a  $GL_h(n) \times GL_{h'}(m)$ -covariant  $(hh')$ -bosonic [or  $(hh')$ -fermionic] algebra.

It is easy to see that the same procedure applied to Eqs. (3.4), (3.5), and (3.7) defining  $\mathcal{A}_{q\sigma}^{(2)}(n, m)$  in the  $\{\mathbf{A}'_{is^+}, \mathbf{A}'_{is}\}$  basis leads to the same equations (4.5)–(4.7) because  $R_h$  and  $\mathcal{R}_{h'}$  are triangular. The algebra  $\mathcal{A}_{hh'\sigma}(n, m)$  is therefore the contraction limit of both  $\mathcal{A}_{q\sigma}^{(1)}(n, m)$  and  $\mathcal{A}_{q\sigma}^{(2)}(n, m)$ .

From Eqs. (4.5) and (4.6), it is clear that contrary to what happens in the  $q$ -deformed case,  $\mathbf{A}_{is}$  can never be considered as the adjoint of  $\mathbf{A}_{is}^+$ . This comes from the lack of  $*$ -structure on  $GL_h(N)$ .

Equations (4.5)–(4.7) agree with the general form of  $\mathcal{H}$ -covariant deformed bosonic (or fermionic) algebras for triangular Hopf algebras  $\mathcal{H}$

which was derived by Fiore (1997). In the present paper, we have established that they can be obtained in a straightforward way by Alishahiha's contraction technique (Alishahiha, 1995).

By using the explicit expression of  $R_h$  given in Eq. (2.8) and a similar one for  $\mathcal{R}_{h'}$ , we can rewrite Eqs. (4.5)–(4.7) in componentwise form as follows:

$$\begin{aligned} [\mathbf{A}_{is}^+, \mathbf{A}_{jt}^+] &= (1 - \sigma P_{ij} P_{st}) \{ h \delta_{j,n} (1 - \delta_{\sigma,-1} \delta_{i,1} \delta_{s,t}) d_i \mathbf{A}_{1s}^+ \mathbf{A}_{it}^+ \\ &\quad + h' \delta_{t,m} (1 - \delta_{\sigma,-1} \delta_{i,j} \delta_{s,1}) \mathbf{d}_s \mathbf{A}_{i1}^+ \mathbf{A}_{js}^+ \\ &\quad - hh' \delta_{j,n} \delta_{t,m} [1 - \delta_{\sigma,-1} (\delta_{i,1} \delta_{s,1} \\ &\quad + \delta_{i,1} \delta_{s,m} + \delta_{i,n} \delta_{s,1})] d_i \mathbf{d}_s \mathbf{A}_{11}^+ \mathbf{A}_{is}^+ \} \end{aligned} \quad (4.9)$$

$$\begin{aligned} [\mathbf{A}_{is}, \mathbf{A}_{jt}] &= -(1 - \sigma P_{ij} P_{st}) \{ h \delta_{j,1} (1 - \delta_{\sigma,-1} \delta_{i,n} \delta_{s,t}) d_i \mathbf{A}_{ns} \mathbf{A}_{it} \\ &\quad + h' \delta_{t,1} (1 - \delta_{\sigma,-1} \delta_{i,j} \delta_{s,m}) \mathbf{d}_s \mathbf{A}_{im} \mathbf{A}_{js} \\ &\quad + hh' \delta_{j,1} \delta_{t,1} [1 - \delta_{\sigma,-1} (\delta_{i,1} \delta_{s,m} \\ &\quad + \delta_{i,n} \delta_{s,1} + \delta_{i,n} \delta_{s,m})] d_i \mathbf{d}_s \mathbf{A}_{nm} \mathbf{A}_{is} \} \end{aligned} \quad (4.10)$$

$$\begin{aligned} [\mathbf{A}_{is}, \mathbf{A}_{jt}^+] &= \delta_{i,j} \delta_{s,t} (\mathbf{I} + \sigma hh' d_i \mathbf{d}_s \mathbf{A}_{11}^+ \mathbf{A}_{nm}) \\ &\quad + \sigma h \delta_{i,j} d_i [\mathbf{A}_{1t}^+ \mathbf{A}_{ns} + h' \delta_{s,1} \delta_{t,m} (-\mathbf{B}_{1n} + h' \mathbf{A}_{11}^+ \mathbf{A}_{nm})] \\ &\quad + \sigma h' \delta_{s,t} \mathbf{d}_s [\mathbf{A}_{j1}^+ \mathbf{A}_{im} + h \delta_{i,1} \delta_{j,n} (-\mathcal{B}_{1m} + h \mathbf{A}_{11}^+ \mathbf{A}_{nm})] \\ &\quad + \sigma h \delta_{i,1} \delta_{j,n} (-\mathcal{B}_{ts} + h \mathbf{A}_{1t}^+ \mathbf{A}_{ns}) + \sigma h' \delta_{s,1} \delta_{t,m} (-\mathbf{B}_{jt} + h' \mathbf{A}_{j1}^+ \mathbf{A}_{im}) \\ &\quad + \sigma hh' \delta_{i,1} \delta_{j,n} \delta_{s,1} \delta_{t,m} (\mathbf{D} - h \mathbf{B}_{1n} - h' \mathcal{B}_{1m} + hh' \mathbf{A}_{11}^+ \mathbf{A}_{nm}) \end{aligned} \quad (4.11)$$

where

$$d_i = 2 - \delta_{i,1} - \delta_{i,n}, \quad \mathbf{d}_s = 2 - \delta_{s,1} - \delta_{s,m} \quad (4.12)$$

$$\mathbf{B}_{ij} = \sum_u \mathbf{d}_u \mathbf{A}_{iu}^+ \mathbf{A}_{ju}, \quad \mathcal{B}_{st} = \sum_k d_k \mathbf{A}_{ks}^+ \mathbf{A}_{kt},$$

$$\mathbf{D} = \sum_{ku} d_k \mathbf{d}_u \mathbf{A}_{ku}^+ \mathbf{A}_{ku} \quad (4.13)$$

and  $P_{ij}$  (resp.  $P_{st}$ ) is the permutation operator acting on  $i, j$  (resp.  $s, t$ ) indices.

In the  $m = 1$  case, Eqs. (4.9)–(4.11) assume a much simpler form:

$$[A_i^+, A_j^+] = (1 - \sigma P_{ij}) [h \delta_{j,n} (1 - \delta_{\sigma,-1} \delta_{i,1}) d_i A_{1s}^+ A_{it}^+] \quad (4.14)$$

$$[A_i, A_j] = -(1 - \sigma P_{ij}) [h \delta_{j,1} (1 - \delta_{\sigma,-1} \delta_{i,n}) d_i A_n A_i] \quad (4.15)$$

$$\begin{aligned}
 [A_i, A_j^\dagger] &= \delta_{i,j}(I + \sigma h d_i A_1^\dagger A_n) \\
 &\quad + \sigma h \delta_{i,1} \delta_{j,n} \left( -\sum_k d_k A_k^\dagger A_k + h A_1^\dagger A_n \right) \quad (4.16)
 \end{aligned}$$

Let us next consider Eqs. (3.4), (3.8), and (3.9), defining  $\mathcal{A}_{q\sigma}^{(1)}(n, m)$  in the  $\{\mathbf{A}'_{is}^\dagger, \tilde{\mathbf{A}}'_{is}\}$  basis. Introducing transformed  $q$ -bosonic (or  $q$ -fermionic) creation operators  $\mathbf{A}''^+ = \mathbf{A}'^+ \mathbf{g}$  as before, and accordingly  $\tilde{\mathbf{A}}'' = \tilde{\mathbf{A}}' \mathbf{g}$ , we notice that compatibility of the  $\tilde{\mathbf{A}}''$  and  $\mathbf{A}''$  definitions with  $\tilde{\mathbf{A}}'' = \mathbf{A}'' \mathbf{C}''$ , where  $\mathbf{C}'' = C''_q \mathcal{C}''_{q\sigma}$ , leads to  $C''_q = g' C'_q g$ , and  $\mathcal{C}''_{q\sigma} = g^s \mathcal{C}'_{q\sigma} g$ . A simple calculation shows that for  $n > 1$

$$\begin{aligned}
 C''_q &= \sum_i (-1)^{n-i} q^{-(n-2i+1)/2} e_{ii'} + \eta(q^{(n-1)/2}) \\
 &\quad + (-1)^{n-1} q^{-(n-1)/2} e_{nn} \quad (4.17)
 \end{aligned}$$

which can be rewritten as

$$\begin{aligned}
 C''_q &= \sum_i (-1)^i q^{-(n-2i-1)/2} e_{ii'} + h(q^{(n-3)/2} + q^{(n-5)/2} + \dots + q^{-(n-1)/2}) e_{nn} \\
 &\quad \text{if } n = 2, 4, \dots \\
 &= \sum_i (-1)^{i-1} q^{-(n-2i+1)/2} e_{ii'} + \eta(q^{(n-1)/2} + q^{-(n-1)/2}) e_{nn} \\
 &\quad \text{if } n = 3, 5, \dots \quad (4.18)
 \end{aligned}$$

We conclude that except for the trivial  $n = 1$  case, wherein we may set  $C'_q = C''_q = C_h = 1$ , a contraction limit of  $C''_q$  only exists for even  $n$  values, and is given by

$$C_h \equiv \lim_{q \rightarrow 1} C''_q = \sum_i (-1)^i e_{ii'} + (n - 1) h e_{nn} \quad (4.19)$$

Similarly, for even  $m$  values,

$$\mathcal{C}_{h'} \equiv \lim_{q \rightarrow 1} \mathcal{C}''_{q\sigma} = \sum_s (-1)^s e_{ss'} + (m - 1) h' e_{mm} \quad (4.20)$$

Restricting the range of  $n, m$  values to  $\{1, 2, 4, 6, \dots\}$ , we obtain that after transformation, Eqs. (3.4), (3.8), and (3.9) contract into

$$\mathbf{A}_1^\dagger \mathbf{A}_2^\dagger = \sigma \mathbf{A}_2^\dagger \mathbf{A}_1^\dagger R_h \mathcal{R}_{h'} \quad (4.21)$$

$$\tilde{\mathbf{A}}_1 \tilde{\mathbf{A}}_2 = \sigma \tilde{\mathbf{A}}_2 \tilde{\mathbf{A}}_1 R_h \mathcal{R}_{h'} \quad (4.22)$$

$$\tilde{\mathbf{A}}_2 \mathbf{A}_1^\dagger = \mathbf{C}_{12} + \sigma \mathbf{A}_1^\dagger \tilde{\mathbf{A}}_2 \tilde{R}_h^{-1} \tilde{\mathcal{R}}_{h'}^{-1} \quad (4.23)$$

where  $\mathbf{C} = C_h \mathcal{C}_{h'}$ ,

$$\begin{aligned} \tilde{R}_h &\equiv \lim_{q \rightarrow 1} (g^{-1} \otimes g^{-1}) \tilde{R}_q (g \otimes g) \\ &= C_{h,1}^{-1} (R_h^{-1})^{t_1} C_{h,1} = C_{h,2}^{-1} (R_h^{t_2})^{-1} C_{h,2} \\ &= \sum_{ij} e_{ii} \otimes e_{jj} - h \sum_t (-1)^i d_i (e_{1i} \otimes e_{1t'} + e_{in} \otimes e_{i'n}) \\ &\quad + (2n - 3) h^2 e_{1n} \otimes e_{1n} \end{aligned} \tag{4.24}$$

and  $\tilde{\mathcal{R}}_{h'}$  is defined in the same way.

Again the same procedure applied to Eqs. (3.4), (3.8), and (3.10), defining  $\mathcal{A}_{q\sigma}^{(2)}(n, m)$  in the  $\{\mathbf{A}'_{is^+}, \tilde{\mathbf{A}}'_{is}\}$  basis, leads to Eqs. (4.21)–(4.23), already obtained for  $\mathcal{A}_{q\sigma}^{(1)}(n, m)$ . We conclude that for  $n, m \in \{1, 2, 4, 6, \dots\}$ , such equations yield another form of the  $\text{GL}_h(n) \times \text{GL}_{h'}(m)$ -covariant  $(hh')$ -bosonic [or  $(hh')$ -fermionic] algebra  $\mathcal{A}_{hh'\sigma}(n, m)$ , defined in Eqs. (4.5)–(4.7) for arbitrary  $n, m$  values. The counterpart of transformation (4.8) is now

$$\varphi(\mathbf{A}^+) = \mathbf{A}^+ \mathcal{T} \tilde{\mathcal{T}}, \quad \varphi(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}} \tilde{\mathcal{T}} \tilde{\mathcal{T}} \tag{4.25}$$

where  $T_{ij} \in \text{GL}_h(n)$ ,  $\mathcal{T}_{st} \in \text{GL}_{h'}(m)$ ,  $\tilde{T} = C_h^{-1} (T^{-1})^t C_h$ , and  $\tilde{\mathcal{T}} = \mathcal{C}_{h'}^{-1} (\mathcal{T}^{-1})^t \mathcal{C}_{h'}$ . However, for  $n$  and/or  $m \in \{3, 5, 7, \dots\}$ , the contraction procedure does not preserve the equivalence between the two forms of  $\mathcal{A}_{q\sigma}^{(1)}(n, m)$  or  $\mathcal{A}_{q\sigma}^{(2)}(n, m)$ , corresponding to the  $\{\mathbf{A}'_{is^+}, \mathbf{A}'_{is}\}$  and  $\{\mathbf{A}'_{is^+}, \tilde{\mathbf{A}}'_{is}\}$  bases, respectively, since only the former has a limit. It should be stressed that such results are entirely new, since Fiore (1997) did not consider any  $\tilde{\mathbf{A}}'_{is}$  operators.

In componentwise form, Eq. (4.21) becomes Eq. (4.9), Eq. (4.22) assumes a similar form, while Eq. (4.23) leads to the following relation:

$$\begin{aligned} \{\tilde{\mathbf{A}}'_{is}, \mathbf{A}'_{jt^+}\} &= \delta_{i',j} \delta_{s',t} (-1)^{i+s} (\mathbf{I} + \sigma h h' d_i \mathbf{d}_s \mathbf{A}'_{11} \tilde{\mathbf{A}}_{11}) \\ &\quad - \delta_{i',j} (-1)^i \{\sigma h d_i \mathbf{A}'_{1t^+} \tilde{\mathbf{A}}_{1s} + h' \delta_{s,m} \delta_{t,m} [(m - 1) \mathbf{I} \\ &\quad + \sigma h d_i \tilde{\mathbf{B}}_{11} + \sigma (2m - 3) h h' d_i \mathbf{A}'_{11} \tilde{\mathbf{A}}_{11}]\} \\ &\quad - \delta_{s',t} (-1)^s \{\sigma h' d_s \mathbf{A}'_{j1} \tilde{\mathbf{A}}_{i1} + h \delta_{i,n} \delta_{j,n} [(n - 1) \mathbf{I} \\ &\quad + \sigma h' d_s \tilde{\mathbf{B}}_{11} + \sigma (2n - 3) h h' d_s \mathbf{A}'_{11} \tilde{\mathbf{A}}_{11}]\} \\ &\quad + \sigma h \delta_{i,n} \delta_{j,n} [\tilde{\mathbf{B}}_{is} + (2n - 3) h \mathbf{A}'_{1t^+} \tilde{\mathbf{A}}_{1s}] \\ &\quad + \sigma h' \delta_{s,m} \delta_{t,m} [\tilde{\mathbf{B}}_{ji} + (2m - 3) h' \mathbf{A}'_{j1} \tilde{\mathbf{A}}_{i1}] \\ &\quad + h h' \delta_{i,n} \delta_{j,n} \delta_{s,m} \delta_{t,m} [(n - 1)(m - 1) \mathbf{I} + \sigma \tilde{\mathbf{D}} + \sigma (2n - 3) h \tilde{\mathbf{B}}_{11} \\ &\quad + \sigma (2m - 3) h' \tilde{\mathbf{B}}_{11} + \sigma (2n - 3)(2m - 3) h h' \mathbf{A}'_{11} \tilde{\mathbf{A}}_{11}] \end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
 \mathbf{B}_{ij} &= \sum_u (-1)^u \mathbf{d}_u \mathbf{A}_{iu}^+ \tilde{\mathbf{A}}_{ju'}, \\
 \mathfrak{B}_{st} &= \sum_k (-1)^k d_k \mathbf{A}_{ks}^+ \tilde{\mathbf{A}}_{kt'}, \\
 \mathbf{D} &= \sum_{ku} (-1)^{k+u} d_k \mathbf{d}_u \mathbf{A}_{ku}^+ \tilde{\mathbf{A}}_{k'u'}
 \end{aligned}
 \tag{4.27}$$

In the  $m = 1$  case, Eq. (4.26) assumes the simpler form

$$\begin{aligned}
 [\tilde{A}_i, A_j^+] &= \delta_{i',j} (-1)^{i+1} (I + \sigma h d_i A_1^+ \tilde{A}_1) + h \delta_{i,n} \delta_{j,n} [(n-1)I \\
 &\quad + \sigma \sum_k (-1)^k d_k A_k^+ \tilde{A}_k + \sigma(2n-3)h A_1^+ \tilde{A}_1]
 \end{aligned}
 \tag{4.28}$$

where  $\tilde{A} = AC_h$ .

In the next section, by making explicit use of the  $U_h(\mathfrak{sl}(2))$  CGC determined by Van der Jeugt (1998), we show that whenever  $n = 2$  and  $m = 1$  or  $2$ , the (anti)commutators (4.9) and (4.26) can be rewritten in coupled form as in the  $q$ -deformed case.

### 5. SPECIAL CASES $n = 2, m = 1$ , AND $n = m = 2$

Let us first consider the  $n = 2, m = 1$  case, wherein

$$R_h = \begin{pmatrix} 1 & h & -h & h^2 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_h = \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix}, \quad \mathfrak{R}_{h'} = \mathfrak{C}_{h'} = 1
 \tag{5.1}$$

From Eqs. (4.14)–(4.16), and (4.28), it follows that the defining relations of the  $GL_h(2)$ -covariant  $h$ -bosonic algebra  $\mathcal{A}_{h+}(2,1)$  are given by

$$[A_1^+, A_2^+] = h(A_1^+)^2, \quad [A_1, A_2] = hA_2^2
 \tag{5.2}$$

$$[A_2, A_1^+] = 0, \quad [A_1, A_2^+] = h(-A_1^+ A_1 - A_2^+ A_2 + hA_1^+ A_2)
 \tag{5.3}$$

$$[A_1, A_1^+] = [A_2, A_2^+] = I + hA_1^+ A_2
 \tag{5.4}$$

in the  $\{A_1^+, A_2^+, A_1, A_2\}$  basis, and by

$$[\tilde{A}_1^+, \tilde{A}_2^+] = h(\tilde{A}_1^+)^2, \quad [\tilde{A}_1, \tilde{A}_2] = h\tilde{A}_1^2
 \tag{5.5}$$

$$[\tilde{A}_1, A_1^+] = 0, \quad [\tilde{A}_2, A_2^+] = h(I - A_1^+ \tilde{A}_2 + A_2^+ \tilde{A}_1 + hA_1^+ \tilde{A}_1)
 \tag{5.6}$$

$$[\tilde{A}_1, A_2^\dagger] = -[\tilde{A}_2, A_1^\dagger] = I + \hbar A_1^\dagger \tilde{A}_1 \quad (5.7)$$

in the  $\{A_1^\dagger, A_2^\dagger, \tilde{A}_1, \tilde{A}_2\}$  basis.

Similarly, for the  $\hbar$ -fermionic algebra  $\mathcal{A}_{\hbar-}(2, 1)$ , we obtain

$$\{A_1^\dagger, A_1^\dagger\} = \{A_1^\dagger, A_2^\dagger\} = 0, \quad \{A_2^\dagger, A_2^\dagger\} = 2\hbar A_1^\dagger A_2^\dagger \quad (5.8)$$

$$\{A_1, A_1\} = 2\hbar A_1 A_2, \quad \{A_1, A_2\} = \{A_2, A_2\} = 0 \quad (5.9)$$

$$\{A_2, A_1^\dagger\} = 0, \quad \{A_1, A_2^\dagger\} = \hbar(A_1^\dagger A_1 + A_2^\dagger A_2 - \hbar A_1^\dagger A_2) \quad (5.10)$$

$$\{A_1, A_1^\dagger\} = \{A_2, A_2^\dagger\} = I - \hbar A_1^\dagger A_2 \quad (5.11)$$

and

$$\{A_1^\dagger, A_1^\dagger\} = \{A_1^\dagger, A_2^\dagger\} = 0, \quad \{A_2^\dagger, A_2^\dagger\} = 2\hbar A_1^\dagger A_2^\dagger \quad (5.12)$$

$$\{\tilde{A}_1, \tilde{A}_1\} = \{\tilde{A}_1, \tilde{A}_2\} = 0, \quad \{\tilde{A}_2, \tilde{A}_2\} = 2\hbar \tilde{A}_1 \tilde{A}_2 \quad (5.13)$$

$$\{\tilde{A}_1, A_1^\dagger\} = 0, \quad \{\tilde{A}_2, A_2^\dagger\} = \hbar(I + A_1^\dagger \tilde{A}_2 - A_2^\dagger \tilde{A}_1 - \hbar A_1^\dagger \tilde{A}_1) \quad (5.14)$$

$$\{\tilde{A}_1, A_2^\dagger\} = -\{\tilde{A}_2, A_1^\dagger\} = I - \hbar A_1^\dagger \tilde{A}_1 \quad (5.15)$$

respectively.

The operators  $(A_1^\dagger, A_2^\dagger)$ , and  $(\tilde{A}_1, \tilde{A}_2)$  may be considered as the components  $m = 1/2$  and  $m = -1/2$  of ITO of rank  $1/2$ , or spinors, with respect to the quantum algebra  $U_\hbar(\mathfrak{sl}(2))$ . By using a nonlinear invertible map between the generators of  $U_\hbar(\mathfrak{sl}(2))$  and  $U(\mathfrak{sl}(2))$  (Abdesselam *et al.*, 1996) and considering the adjoint action of the former on such spinors, Aizawa (1998) recently realized them in terms of standard bosonic or fermionic operators  $a_1^\dagger, a_2^\dagger, a_1, a_2$ . For the standard form of  $\mathfrak{sl}(2)$  generators

$$J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_0 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \quad (5.16)$$

the realizations read<sup>2</sup>

$$A_1^\dagger = \left(1 - \frac{\hbar}{2} J_+\right)^{-1} a_1^\dagger, \quad A_2^\dagger = \left(1 - \frac{\hbar}{2} J_+\right)^{-1} a_2^\dagger + \frac{\hbar}{2} (A_1^\dagger - 2a_1^\dagger J_0) \quad (5.17)$$

$$\tilde{A}_1 = \left(1 - \frac{\hbar}{2} J_+\right)^{-1} a_2, \quad \tilde{A}_2 = -\left(1 - \frac{\hbar}{2} J_+\right)^{-1} a_1 + \frac{\hbar}{2} (\tilde{A}_1 - 2a_2 J_0) \quad (5.18)$$

<sup>2</sup>The realization of  $\mathfrak{sl}(2)$  used in Eqs. (5.19) and (5.20) differs from that considered by Aizawa (1998). There are also some changes of phase conventions with respect to that reference in Eqs. (5.17)–(5.20).

in the  $h$ -bosonic case, and

$$A_1^+ = a_1^+, \quad A_2^+ = a_2^+ - 2ha_1^+J_0 \tag{5.19}$$

$$\tilde{A}_1 = a_2, \quad \tilde{A}_2 = -a_1 - 2ha_2J_0 \tag{5.20}$$

in the  $h$ -fermionic one. As expected, the operators (5.17), (5.18) and (5.19), (5.20) satisfy Eqs. (5.5)–(5.7) and (5.12)–(5.15), respectively.

Let us now introduce coupled (anti)commutators, defined as in Eq. (3.14) by

$$[T^{j_1}, U^{j_2}]_M^J = [T^{j_1} \times U^{j_2}]_M^J - \sigma(-1)^\epsilon [U^{j_2} \times T^{j_1}]_M^J \tag{5.21}$$

Here  $T^{j_1}$  and  $U^{j_2}$  denote two ITOs of rank  $j_1$  and  $j_2$  with respect to  $U_h(\mathfrak{sl}(2))$ , respectively,  $\epsilon$  is defined as in Eqs. (3.16) and (3.17) by  $\epsilon = j_1 + j_2 - J$ , and

$$[T^{j_1} \times U^{j_2}]_M^J = \sum_{m_1 m_2} \langle j_1 m_1, j_2 m_2 | JM \rangle_h T_{m_1}^{j_1} U_{m_2}^{j_2} \tag{5.22}$$

where  $\langle, \rangle_h$  denotes a  $U_h(\mathfrak{sl}(2))$  CGC (Van der Jeugt, 1998). The values of the latter needed for coupling spinors are given in Table I. By using them, we can recast Eqs. (5.5)–(5.7) and (5.12)–(5.15) in the compact forms

$$[A^+, A^+]_0^0 = [\tilde{A}, \tilde{A}]_0^0 = [\tilde{A}, A^+]_M^0 = 0, \quad [\tilde{A}, A^+]_0^0 = \sqrt{2}I \tag{5.23}$$

and

$$\{A^+, A^+\}_M^1 = \{\tilde{A}, \tilde{A}\}_M^1 = \{\tilde{A}, A^+\}_M^1 = 0, \quad \{\tilde{A}, A^+\}_0^0 = \sqrt{2}I \tag{5.24}$$

respectively.

Let us next consider the  $n = m = 2$  case, wherein  $\mathcal{R}_{h'}$  and  $\mathcal{C}_{h'}$  are defined as  $R_h$  and  $C_h$  in Eq. (5.1). Relations similar to Eqs. (5.2)–(5.15) can be easily written. The operators  $\mathbf{A}_{is}^+$  ( $i, s = 1, 2$ ) and  $\tilde{\mathbf{A}}_{is}$  ( $i, s = 1, 2$ ) may now be considered as the components of double spinors with respect to  $U_h(\mathfrak{sl}(2)) \times U_{h'}(\mathfrak{sl}(2))$ . Defining coupled (anti)commutators by

$$[T^{j_1 j'_1}, U^{j_2 j'_2}]_{MM'}^{JJ'} = [T^{j_1 j'_1} \times U^{j_2 j'_2}]_{MM'}^{JJ'} - \sigma(-1)^\epsilon [U^{j_2 j'_2} \times T^{j_1 j'_1}]_{MM'}^{JJ'} \tag{5.25}$$

**Table I.** Values of  $U_h(\mathfrak{sl}(2))$  CGC  $\langle \frac{1}{2} m_1, \frac{1}{2} m_2 | JM \rangle_h$

	$J = M = 1$	$J = 1, M = 0$	$J = -M = 1$	$J = M = 0$
$m_1 = m_2 = 1/2$	1	0	$(h/2)^2$	$-h/\sqrt{2}$
$m_1 = -m_2 = 1/2$	0	$1/\sqrt{2}$	$-h/2$	$1/\sqrt{2}$
$m_1 = -m_2 = -1/2$	0	$1/\sqrt{2}$	$h/2$	$-1/\sqrt{2}$
$m_1 = m_2 = -1/2$	0	0	1	0

where  $\epsilon = j_1 + j_2 - J + j'_1 + j'_2 - J'$ , and

$$\begin{aligned}
 & [T^{j_1 j'_1} \times U^{j_2 j'_2}]_{MM'}^{JJ'} \\
 &= \sum_{m_1 m_2 m'_1 m'_2} \langle j_1 m_1, j_2 m_2 | JM \rangle_h \langle j'_1 m'_1, j'_2 m'_2 | J' M' \rangle_{h'} T_{m_1 m'_1}^{j_1 j'_1} U_{m_2 m'_2}^{j_2 j'_2} \quad (5.26)
 \end{aligned}$$

we easily obtain that the double spinors  $\mathbf{A}^+$  and  $\tilde{\mathbf{A}}$  satisfy the relations

$$[\mathbf{A}^+, \mathbf{A}^+]_{M0}^{10} = [\mathbf{A}^+, \mathbf{A}^+]_{0M'}^{01} = [\tilde{\mathbf{A}}, \tilde{\mathbf{A}}]_{M0}^{10} = [\tilde{\mathbf{A}}, \tilde{\mathbf{A}}]_{0M'}^{01} = 0 \quad (5.27)$$

$$[\tilde{\mathbf{A}}, \mathbf{A}^+]_{MM'}^{JJ'} = 2\delta_{J,0}\delta_{J',0}\delta_{M,0}\delta_{M',0}\mathbf{I} \quad (5.28)$$

and

$$\{\mathbf{A}^+, \mathbf{A}^+\}_{MM'}^{11} = \{\mathbf{A}^+, \mathbf{A}^+\}_{00}^{00} = \{\tilde{\mathbf{A}}, \tilde{\mathbf{A}}\}_{MM'}^{11} = \{\tilde{\mathbf{A}}, \tilde{\mathbf{A}}\}_{00}^{00} = 0 \quad (5.29)$$

$$\{\tilde{\mathbf{A}}, \mathbf{A}^+\}_{MM'}^{JJ'} = 2\delta_{J,0}\delta_{J',0}\delta_{M,0}\delta_{M',0}\mathbf{I} \quad (5.30)$$

in the  $(hh')$ -bosonic and  $(hh')$ -fermionic cases, respectively.

It is remarkable that Eqs. (5.23) [resp. (5.24)] and (5.27), (5.28) [resp. (5.29), (5.30)] are formally identical to those for bosonic (resp. fermionic) ITOs with respect to the Lie algebras  $\mathfrak{sl}(2)$  and  $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$ , respectively. Contrary to what happens in the  $q$ -bosonic (or  $q$ -fermionic) case, where the (anti)commutators are  $q$ -deformed [see Eqs. (3.22) and (3.23)], here all the dependence upon the deforming parameters  $h, h'$  is contained in the coupling coefficients.

## 6. CONCLUSION

In the present paper, we showed that the contraction technique previously used to construct Jordanian deformations of Lie groups from standard ones (Aghamohammadi *et al.*, 1995; Alishahiha, 1995) can be applied to the  $\text{GL}_q(n) \times \text{GL}_q(m)$ -covariant  $q$ -bosonic [or  $\text{GL}_q(n) \times \text{GL}_q^{-1}(m)$ -covariant  $q$ -fermionic] algebras  $\mathcal{A}_{q\pm}^{(\alpha)}(n, m)$ ,  $\alpha = 1, 2$  (Quesne, 1993, 1994; Fiore, 1998) to yield  $\text{GL}_h(n) \times \text{GL}_{h'}(m)$ -covariant  $(hh')$ -bosonic [or  $(hh')$ -fermionic] algebras  $\mathcal{A}_{hh'\pm}(n, m)$ . In this process, the arbitrariness present in the  $q$ -deformed case disappears, as the algebras  $\mathcal{A}_{q\pm}^{(1)}(n, m)$  and  $\mathcal{A}_{q\pm}^{(2)}(n, m)$  have the same contraction limit  $\mathcal{A}_{hh'\pm}(n, m)$ .

When using a basis  $\{\mathbf{A}'_{is}, \mathbf{A}'_{is}\}$  of  $\mathcal{A}_{q\pm}^{(\alpha)}(n, m)$ , wherein the annihilation operators  $\mathbf{A}'_{is}$  are contragredient to the creation ones  $\mathbf{A}'_{is}$ , this contraction procedure can be carried out for any  $n, m$  values. The resulting defining relations of  $\mathcal{A}_{hh'\pm}(n, m)$  were written in the contracted basis  $\{\mathbf{A}'_{is}, \mathbf{A}_{is}\}$ , both in compact form in terms of the defining  $R_h$  and  $\mathcal{R}_{h'}$ -matrices of  $\text{GL}_h(n)$  and  $\text{GL}_{h'}(m)$ , respectively, and in componentwise form. They may be considered as a special case of the defining relations of  $\mathcal{H}$ -covariant deformed bosonic

(or fermionic) algebras for triangular Hopf algebras  $\mathcal{H}$ , recently obtained by Fiore (1997) by another procedure.

When using instead a basis  $\{\mathbf{A}'_{is^+}, \tilde{\mathbf{A}}'_{is}\}$  of  $\mathcal{A}_{q^\pm}^{(\alpha)}(n, m)$ , wherein the annihilation operators  $\tilde{\mathbf{A}}'_{is}$  are ITOs with respect to the quantum algebra  $U_q(\mathfrak{gl}(n)) \times U_{q^\pm 1}(\mathfrak{gl}(m))$ , we obtained some new and interesting results. We established that in such a case a contraction limit only exists whenever  $n, m \in \{1, 2, 4, 6, \dots\}$ , hence showing that for  $n$  and/or  $m \in \{3, 5, 7, \dots\}$ , the contraction procedure does not preserve the equivalence between the two forms of  $\mathcal{A}_{q^\pm}^{(\alpha)}(n, m)$ , corresponding to the  $\{\mathbf{A}'_{is^+}, \mathbf{A}'_{is}\}$  and  $\{\mathbf{A}'_{is^+}, \tilde{\mathbf{A}}'_{is}\}$  bases. When a limit does exist, the defining relations of  $\mathcal{A}_{hh'\pm}(n, m)$  were written in the contracted basis  $\{\mathbf{A}'_{is^+}, \tilde{\mathbf{A}}'_{is}\}$  both in compact form in terms of  $R_h$  and  $\mathcal{R}_{h'}$  and in componentwise form.

Such a basis is essential to express the defining relations of  $\mathcal{A}_{hh'\pm}(n, m)$  in another compact form in terms of coupled (anti)commutators, thereby enhancing the transformation properties of the generators under the quantum algebra dual to  $GL_h(n) \times GL_{h'}(m)$ . We proved this point in the  $n = 2, m = 1$ , and  $n = m = 2$  cases, where the dual quantum algebras are known, and the  $U_h(\mathfrak{sl}(2))$  CGC determined by Van der Jeugt (1998) can be used. Furthermore, we checked that the  $h$ -bosonic and  $h$ -fermionic ITOs of rank  $1/2$  with respect to  $U_h(\mathfrak{sl}(2))$  constructed by Aizawa (1998) satisfy the defining relations of  $\mathcal{A}_{hh'\pm}(2, 1)$ . From the examples considered, we concluded that the algebras  $\mathcal{A}_{hh'\pm}(n, m)$  are much closer to the standard Heisenberg (or Clifford) algebras  $\mathcal{A}_\pm(n, m)$  than the  $q$ -deformed ones,  $\mathcal{A}_{q^\pm}^{(\alpha)}(n, m)$ . This may be an advantage in some physical applications.

## ACKNOWLEDGMENT

The author is a Research Director of the National Fund for Scientific Research (FNRS), Belgium.

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